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FINITE STRETCHING OF AN ANNULAR PLATE†

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Abstract—The problem of the finite stretching of an annular plate which is bonded to a rigid inclusion at its inner edge is considered. The material is assumed to be isotropic and incompressible with a Mooney-type constitutive law. It is shown that the inclusion of the effect of the transverse normal strain leads to a rapid variation in thickness which is confined to a narrow edge zone. The explicit solutions to the boundary layer equations, which govern the behavior of the plate near the edges, are presented.

NOTATION

ξ, θ	plane polar coordinates on the undeformed middle plane of the plate
z	coordinate along the normal of the middle plane
$\alpha_\xi, \alpha_\theta^*$	coefficients of the first fundamental form of the deformed plate $\alpha_\xi^2\, d\xi^2 + \alpha_\theta^{*2}\, d\theta^2$
$\lambda_\xi, \lambda_\theta, \lambda$	principal extension ratios [defined by (1)] in ξ, θ and z directions
u^*	radial displacement
h_0	thickness of the undeformed plate
$h = \lambda h_0$	thickness of the deformed plate
$p_0 = p_0(\xi)$	hydrostatic pressure
N_ξ, N_θ	membrane stress resultants
$N = \int_{-h/2}^{h/2} \sigma_z\, dz$	average normal stress
$S_\xi = \int_{-h/2}^{h/2} z\sigma_{\xi z}\, dz$	transverse shear stress couple
α, β	coefficients of the Mooney strain energy form
Γ	β/α

INTRODUCTION

THE problem of the strain distribution around a central circular hole in a circular sheet, made of isotropic, incompressible material, and subjected to a uniform radial tension at its outer edge, was first investigated by Rivlin and Thomas [1]. In this paper, the problem is formulated using the principal values of the Cauchy–Green deformation tensor, which are called the principal extensions, and solved numerically for a Mooney material. The same problem also is included in chapter 4 of Green and Adkins [2], who extended the formulation to transversely isotropic materials. Recently, the related problem of the stress concentration for a circular sheet has been studied by Yang [3], who also presented the results for the case of a circular rigid inclusion. In all of these papers, the plane stress assumption is used, which excludes the effect of the transverse normal strain. However, after the problem

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is solved, a thickness change is calculated, so that the incompressibility condition throughout the plate is satisfied.

The inclusion of the effect of symmetric transverse normal strain, in the context of thin shell theory, has been recently considered by Biricikoglu and Kalnins [4]. The set of equations presented in [4] are directly applicable to the axially symmetric stretching of a circular plate. These equations admit the prescription of a definite symmetric thickness change on the boundaries and are capable of predicting the thickness change throughout the plate. In the case of the stretching problem of a plate, the bending moments and curvatures vanish identically, so that the deformation is truly symmetrical about the middle plane of the plate.

The purpose of this paper is to study the axisymmetric stretching of an annular plate, made of an isotropic and incompressible Mooney-type material which is bonded to a rigid inclusion at its inner edge, so that the thickness of the deformed plate at the inner edge is constrained to be equal to the undeformed thickness of the plate.

The contribution of the present paper is that it includes the effect of the symmetric transverse normal strain and hence it allows the prescription of a definite symmetric thickness change at the edges of the plate.

GOVERNING EQUATIONS

The following set of equations which govern the axisymmetric stretching of a circular plate made of an isotropic, incompressible Mooney material are taken from [4, equations (44), (63), (67) and (68)]. For axisymmetric deformation of a circular plate, the radial and circumferential directions coincide with the principal directions of the Cauchy–Green deformation tensor on the middle plane. The governing equations will be written in terms of the principal values of the Cauchy–Green deformation tensor whose physical components are called the principal extension ratios and are given by

$$\lambda_\xi = \alpha_\xi \tag{1a}$$

$$\lambda_\theta = \alpha_\theta^*/\xi. \tag{1b}$$

The incompressibility condition is

$$\lambda_\xi \lambda_\theta \lambda = 1. \tag{2}$$

The equations of equilibrium are

$$(\alpha_\theta^* N_\xi)_{,\xi} - \alpha_\xi N_\theta = 0 \tag{3a}$$

$$(\alpha_\theta^* S_\xi)_{,\xi} - \alpha_\xi \alpha_\theta^* N = 0 \tag{3b}$$

$$N_\xi = \lambda h_0[-p_0 + 2\alpha\lambda_\xi^2 - 2\beta(1/\lambda_\xi^2 + h_0^2\lambda_\theta^2\lambda_{,\xi}^2/12)] \tag{4a}$$

$$N_\theta = \lambda h_0[-p_0 + 2\alpha\lambda_\theta^2 - 2\beta/\lambda_\theta^2] \tag{4b}$$

$$N = \lambda h_0[-p_0 + 2\alpha(\lambda^2 + h_0^2\lambda_{,\xi}^2/12) - 2\beta/\lambda^2] \tag{4c}$$

$$S_\xi = (h_0^3/12)\lambda_\xi\lambda^2\lambda_{,\xi}[2\alpha + 2\beta\lambda_\theta^2]. \tag{4d}$$

The compatibility condition is

$$\alpha_{\theta,\xi}^* = \alpha_\xi. \tag{5}$$

The definition of the radial displacement u gives

$$\alpha_\theta^* = \xi + u^*. \tag{6}$$

The nonzero component of the rotation vector is

$$\beta_3 = 1 - 1/\lambda. \tag{7}$$

Along a circular edge ξ = const., the natural boundary conditions are that

$$\text{either } N_\xi \text{ or } u^*, \tag{8a}$$

$$\text{either } S_\xi \text{ or } \beta_3 \tag{8b}$$

be prescribed.

SCALED EQUATIONS

The governing equations (1)–(7) predict the axisymmetric stretching of a circular plate with large elastic strains and a symmetric thickness change with respect to the middle plane of the plate. These equations constitute a system of ordinary nonlinear differential equations whose analytical solutions are not easily accessible in terms of simple functions. A close look, however, reveals that (1)–(7) contain terms which are small over most of the plate. This suggests for their solution the introduction of the coordinate stretching technique, which is widely used in boundary layer analyses. With the aid of the coordinate stretching, (1)–(7) can be divided into two distinct groups: the outer problem which predicts the behavior of the plate away from the edges, and the inner problem (boundary layer) which governs the solution near the edges of the plate. The solution to the outer problem, together with the solution in the boundary layer, provides a uniformly valid first approximation to the solution of the system of equations (1)–(7).

In the following, we first introduce the nondimensional independent variable x by

$$x = \xi/L \tag{9}$$

where L is a characteristic length of the deformation pattern. Next, we nondimensionalize the dependent variables by

$$n_x = N_\xi/2\alpha h_0 \qquad n_\theta = N_\theta/2\alpha h_0 \qquad n = N/2\alpha h_0 \tag{10a}$$

$$s_x = (12)^{\frac{1}{2}} S_\xi 2\alpha h_0^2 \tag{10b}$$

$$p = p_0/2\alpha \tag{10c}$$

$$u = u^*/L \qquad \alpha_\theta = \alpha_\theta^*/L \qquad \lambda_x = \lambda_\xi. \tag{10d}$$

The governing equations (1)–(7) then become

$$\alpha_\theta n_{x,x} + \lambda_x(n_x - n_\theta) = 0 \tag{11a}$$

$$\varepsilon(\alpha_\theta s_x)_{,x} - \alpha_\theta \lambda_x n = 0 \tag{11b}$$

$$\lambda_\theta = \alpha_\theta/x \tag{12a}$$

$$u = \alpha_\theta - x \tag{12b}$$

$$\alpha_{\theta,x} = \lambda_x \tag{12c}$$

$$\lambda_x \lambda_\theta \lambda = 1 \tag{13}$$

and

$$n_x = \lambda[-p+\lambda_x^2-\Gamma\lambda_\theta^2(\lambda^2+\varepsilon^2\lambda_{,x}^2)] \tag{14a}$$

$$n_\theta = \lambda[-p+\lambda_\theta^2-\Gamma\lambda_x^2\lambda^2] \tag{14b}$$

$$n = \lambda[-p+\lambda^2-\Gamma\lambda_x^2\lambda_\theta^2+\varepsilon^2\lambda_{,x}^2] \tag{14c}$$

$$s_x = \varepsilon\lambda_x\lambda^2\lambda_{,x}(1+\Gamma\lambda_\theta^2) \tag{14d}$$

where

$$\varepsilon = h_0/(12)^{\frac{1}{2}}L \tag{15}$$

is a nondimensional parameter.

We now suppose that the parameter ε is small compared with unity so that the terms of order ε and smaller can be neglected in (11)–(14). This process yields the following set of equations which govern the deformation of the plate away from the edges.

$$\alpha_\theta n_{x,x}+\lambda_x(n_x-n_\theta) = 0 \tag{16a}$$

$$n = 0 \tag{16b}$$

$$n_x = \lambda(-p+\lambda_x^2-\Gamma\lambda_\theta^2\lambda^2) \tag{17a}$$

$$n_\theta = \lambda(-p+\lambda_\theta^2-\Gamma\lambda_x^2\lambda^2) \tag{17b}$$

$$n = \lambda(-p+\lambda^2-\Gamma\lambda_x^2\lambda_\theta^2) \tag{17c}$$

$$s_x = 0. \tag{17d}$$

The remaining equations (12) and (13) are unchanged. Equation (16b), together with (17c), serves for the determination of the unknown hydrostatic pressure p in terms of the extension ratios.

Since the order of the governing differential equations is reduced by two, the solution to the outer problem is not uniformly valid throughout the entire plate. This necessitates the formation of the boundary layers near the edges of the plate. The proper boundary conditions to the outer problem should be obtained from the matching requirements, so that they will be given after the analysis of the boundary layers is completed.

To study the boundary layer, let $x = x_0$ be the equation of the edge of the plate. The stretched boundary layer coordinate is defined by

$$\tau = (x-x_0)/\varepsilon. \tag{18}$$

In terms of the stretched coordinate τ, the governing equations (11)–(14) become

$$\alpha_\theta n_{x,\tau}+\varepsilon\lambda_x(n_x-n_\theta) = 0 \tag{19a}$$

$$\alpha_\theta s_{x,\tau}+\varepsilon\lambda_x s_x-\alpha_\theta\lambda_x n = 0 \tag{19b}$$

$$\lambda_\theta = \alpha_\theta/(x_0+\varepsilon\tau) \tag{20a}$$

$$u = \alpha_\theta-(x_0+\varepsilon\tau) \tag{20b}$$

$$\alpha_{\theta,\tau} = \varepsilon\lambda_x \tag{20c}$$

$$\lambda_x\lambda_\theta\lambda = 1 \tag{21}$$

$$n_x = \lambda[-p+\lambda_x^2-\Gamma\lambda_\theta^2(\lambda^2+\lambda_{,\tau}^2)] \tag{22a}$$

$$n_\theta = \lambda[-p+\lambda_\theta^2-\Gamma\lambda_x^2\lambda^2] \tag{22b}$$

$$n = \lambda[-p+\lambda^2+\lambda_{,\tau}^2-\Gamma\lambda_x^2\lambda_\theta^2] \tag{22c}$$

$$s_x = \lambda_x\lambda^2\lambda_{,\tau}[1+\Gamma\lambda_\theta^2]. \tag{22d}$$

The boundary layer equations, then, are obtained from (19) to (22) by letting $\varepsilon \to 0$ while keeping τ fixed, which leads to

$$\alpha_{\theta,\tau} = 0 \qquad \alpha_\theta = \delta x_0 = \text{const.} \tag{23a}$$

$$\lambda_\theta = \delta = \text{const.} \tag{23b}$$

$$u = (\delta-1)x_0 = \text{const.} \tag{23c}$$

$$n_{x,\tau} = 0 \qquad n_x = n_0 = \text{const.} \tag{24a}$$

$$s_{x,\tau}-\lambda_x n = 0 \tag{24b}$$

$$n_0 = \lambda[-p+\lambda_x^2-\Gamma\delta^2(\lambda^2+\lambda_{,\tau}^2)] \tag{25a}$$

$$n_\theta = \lambda(-p+\delta^2-\Gamma/\delta^2) \tag{25b}$$

$$n = \lambda(-p+\lambda^2+\lambda_{,\tau}^2-\Gamma\delta^2\lambda_x^2) \tag{25c}$$

$$s_x = \lambda\lambda_{,\tau}(1+\Gamma\delta^2)/\delta \tag{25d}$$

$$\lambda_x\lambda = 1/\delta. \tag{26}$$

Now, (25a) serves for the elimination of the unknown hydrostatic pressure p. The solution to the boundary layer equations (23)–(26) must satisfy the prescribed conditions at the edge of the plate as $\tau \to 0$, and must match the outer solution as $\tau \to \infty$. Since n_x and u are constant throughout the boundary layer, and since the boundary layer solution must match the outer solution as $\tau \to \infty$, it follows that the natural boundary condition for the outer problem is that

$$\text{at } x = x_0 \quad \text{either } n_x \text{ or } u \tag{27}$$

be prescribed.

OUTER PROBLEM

The outer problem is governed by (12), (13), (16) and (17), with the boundary condition (27). Although this problem has been already solved in [1, 3], we consider its solution again, because we need it for the matching with the inner solution and also because we can propose a more systematic method for its solution than that used in [1].

The solutions presented by Rivlin and Thomas [1], and later by Yang [3], are obtained through a direct integration process which requires the boundary values of λ_x, and λ_θ (or equivalently of n_x and u) at the starting point. In this sense, they presented solutions to the corresponding initial value problem rather than the boundary value problem. For the solution of the outer problem, we use the multisegment method of direct numerical integration [5]. According to this method, the boundary value problem is formulated in terms of two variables, called as the fundamental variables, which enter into the boundary conditions of the problem. The multisegment method requires the evaluation of the derivatives of the fundamental variables with respect to the radial coordinate x at any point on the middle plane, when the fundamental variables themselves are known. The calculation of the derivatives is carried out after arranging the system of governing equations (12), (13), (16) and (17) in a certain sequence, namely

$$\alpha_\theta = x+u \tag{28a}$$

$$\lambda_\theta = \alpha_\theta/x \tag{28b}$$

$$\lambda_\theta^2\lambda_x-\frac{1}{\lambda_x^2}-\frac{n_x\lambda_\theta^3}{(1+\Gamma\lambda_\theta^2)} = 0 \tag{28c}$$

$$\lambda = 1/\lambda_x\lambda_\theta \tag{28d}$$

$$n_\theta = \lambda(\lambda_\theta^2-\lambda^2)(1+\Gamma\lambda_x^2) \tag{28e}$$

$$u_{,x} = \lambda_x-1 \tag{28f}$$

$$n_{x,x} = -\lambda_x(n_x-n_\theta)/\alpha_\theta \tag{28g}$$

where (28c) should be solved for λ_x. In the above equations the hydrostatic pressure is eliminated through the use of (16b) and (17c).

The outer boundary value problem is described by (28) with the boundary condition (27).

INNER PROBLEM

The behavior of the plate near the edges is governed by the boundary layer equations (23)–(26). Since n_x and λ_θ are constant in the boundary layer, we can eliminate hydrostatic pressure p using the constitutive relation for n_x, namely, (25a). This yields

$$n = n_0+\lambda(1+\Gamma\delta^2)(\lambda^2+\lambda_{,\tau}^2-\lambda_x^2). \tag{29}$$

Next, we eliminate the radial extension ratio λ_x from (24b) and (29) using the incompressibility condition (26). We then substitute n and s_x into the equilibrium equation (24b) and get a second order nonlinear differential equation for the extension ratio λ

$$\lambda_{,\tau\tau} = \lambda+\frac{n_0}{(1+\Gamma\delta^2)\lambda^2}-\frac{1}{\delta^2\lambda^3}. \tag{30}$$

Since the independent variable is absent on the right hand side, (30) can be integrated to give

$$\lambda_{,\tau}^2 = \lambda^2-\frac{2n_0}{(1+\Gamma\delta^2)\lambda}+\frac{1}{\delta^2\lambda^2}+2C$$

where C is an integration constant. In order to match the outer solution as $\tau \to \infty$, λ must be a monotonically decreasing function of τ in the boundary layer. Hence the derivative of λ must be negative and

$$\lambda_{,\tau} = -\left[\lambda^2+2C-\frac{2n_0}{(1+\Gamma\delta^2)\lambda}+\frac{1}{\delta^2\lambda^2}\right]^{\frac{1}{2}} \tag{31}$$

Substituting (31) into (25d) we get the shear moment s_x as a function of λ

$$s_x = -\frac{(1+\Gamma\delta^2)}{\delta}\left[\lambda^4+2C\lambda^2-\frac{2n_0}{(1+\Gamma\delta^2)}\lambda+\frac{1}{\delta^2}\right]^{\frac{1}{2}}. \tag{32}$$

The constant C is determined through matching with the outer solution. The matching conditions are given by

$$\text{inner } \lambda(\tau \to \infty) = \text{outer } \lambda(x \to x_0) = \sigma \tag{33a}$$

$$\text{inner } s_x(\tau \to \infty) = \text{outer } s_x(x \to x_0) = 0 \tag{33b}$$

where σ is known from the outer solution. Combining (33a) and (33b) we find that

$$\text{inner } s_x(\lambda \to \sigma) = 0 \tag{34}$$

which is the appropriate matching condition since we have an explicit relation between s_x and λ. Using (34), we get from (32) that

$$2C = -\sigma^2+\frac{2n_0}{(1+\Gamma\delta^2)\sigma}-\frac{1}{\delta^2\sigma^2} \tag{35a}$$

so that (32) reduces to

$$s_x = -\frac{(1+\Gamma\delta^2)}{\delta}(\lambda-\sigma)\left(\lambda^2+2\sigma\lambda+\frac{1}{\delta^2\sigma^2}\right)^{\frac{1}{2}} \tag{35b}$$

where we have used

$$n_0 = \sigma\left(\frac{1}{\delta^2\sigma^2}-\sigma^2\right)(1+\Gamma\delta^2) \tag{36}$$

which is obtained from (17a) by eliminating p with the aid of (16b) and (17c).

Let the value of λ at the edge x_0 be denoted by σ^*. Using separation of variables and (35a), (31) leads to

$$\tau = \int_{\lambda}^{\sigma^*}\frac{y\,\mathrm{d}y}{(y-\sigma)[y^2+2\sigma y+(1/\delta^2\sigma^2)]^{\frac{1}{2}}} \tag{37a}$$

which can easily be integrated to give

$$\tau = \tau(\lambda, \sigma, \sigma^*) \tag{37b}$$

in terms of simple functions. This form, however, is not useful because the inversion of (37b) into the form

$$\lambda = \lambda(\tau, \sigma, \sigma^*)$$

is rather difficult. Hence, we prefer to calculate the initial value of s_x by

$$\begin{aligned} s_x^* &= s_x(\tau = 0) \\ &= -\frac{(1+\Gamma\delta^2)}{\delta}(\sigma^*-\sigma)\left(\sigma^{*2}+2\sigma\sigma^*+\frac{1}{\delta^2\sigma^2}\right)^{\frac{1}{2}} \end{aligned} \tag{38}$$

and then integrate the boundary layer equations (24b) and (25b) numerically, since now the problem is actually reduced to an initial value problem.

One interesting remark can be made by observing the form of (35b). If $s_x = 0$ is prescribed at the edge of the plate, (35b) gives that

$$\sigma^* = \sigma$$

which means that the initial value of λ is equal to its outer counterpart. Hence, it follows that for the case

$$s_x(\tau = 0) = 0$$

there is no boundary layer of thickness $0(\varepsilon)$. This case then can be represented by the outer solution alone, which is the case treated by Rivlin and Thomas [1].

If a circular annular sheet is bonded to a rigid inclusion at its inner edge and is subjected to a uniform radial stretching, the boundary conditions at the inner edge are

$$\text{at } x = x_0 \qquad u = 0\,(\text{outer}) \tag{39a}$$

$$\text{at } \tau = 0 \qquad \lambda = \sigma^* = 1\,(\text{inner}). \tag{39b}$$

Then (23c) yields

$$\delta = 1$$

and hence the initial condition of s_x becomes

$$s_x(\tau = 0) = -(1+\Gamma)(1-\sigma)\left(1+2\sigma+\frac{1}{\sigma^2}\right)^{\frac{1}{2}}$$

which then can be used to initiate the initial value integration.

NUMERICAL RESULTS

The solutions of two distinct problems are presented. First, the solution state of an annular circular plate which is subjected to a uniform tension at the outer edge is given (Fig. 1). This problem is solved by the multisegment method of direct numerical integration [5]. The linear solution is taken as the first trial solution, and acceptable convergence is achieved after three iterations. A good indication of convergence is provided by the value of n_x at the outer edge which is supposed to converge to the prescribed value (Table 1). In this problem, since $s_x = 0$ prescribed at the edges, there are no boundary layers of order ε so that the outer solution is assumed to represent the solution state throughout the entire

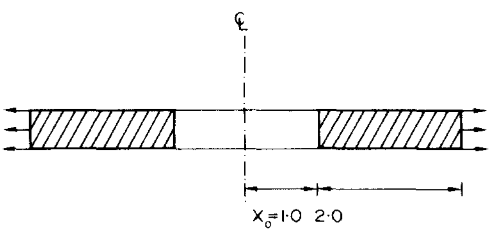

FIG. 1. Section of the annular plate.

TABLE 1. VALUES OF n_x AT $x = 3$

Linear		0·35000
Nonlinear	First iteration	0·27424
	Second iteration	0·33978
	Third iteration	0·34931
	Fourth iteration	0·34996
	Fifth iteration	0·34999

TABLE 2. RADIAL DISPLACEMENT AND RADIAL STRESS RESULTANT FOR FIVE ITERATIONS (MOONEY MATERIAL $\Gamma = 0·1$)

x	u	n_x
1·00	0·3303	0·0000
1·20	0·3094	0·1018
1·40	0·2974	0·1713
1·60	0·2915	0·2204
1·80	0·2900	0·2561
2·00	0·2917	0·2828
2·20	0·2958	0·3031
2·40	0·3017	0·3190
2·60	0·3091	0·3316
2·80	0·3176	0·3417
3·00	0·3271	0·3500

TABLE 3. BONDED PLATE OUTER SOLUTION: RADIAL DISPLACEMENT AND STRESS RESULTANTS (MOONEY MATERIAL $\Gamma = 0·1$)

x	u	n_x	n_θ
1·00	0·0000	1·3160	0·4523
1·20	0·0892	1·1363	0·5677
1·40	0·1650	1·0425	0·6362
1·60	0·2330	0·9866	0·6805
1·80	0·2961	0·9502	0·7109
2·00	0·3557	0·9251	0·7327
2·20	0·4129	0·9070	0·7488
2·40	0·4683	0·8934	0·7610
2·60	0·5224	0·8831	0·7706
2·80	0·5754	0·8749	0·7782
3·00	0·6275	0·8684	0·7843

TABLE 4. BONDED PLATE OUTER SOLUTION: EXTENSION RATIOS (MOONEY MATERIAL $\Gamma = 0·1$)

x	λ_x	λ_θ	λ
1·00	1·4954	1·0000	0·6687
1·20	1·4061	1·0743	0·6620
1·40	1·3565	1·1179	0·6594
1·60	1·3260	1·1457	0·6583
1·80	1·3057	1·1645	0·6577
2·00	1·2915	1·1779	0·6574
2·20	1·2811	1·1877	0·6572
2·40	1·2734	1·1951	0·6571
2·60	1·2674	1·2009	0·6570
2·80	1·2627	1·2055	0·6570
3·00	1·2589	1·2092	0·6569

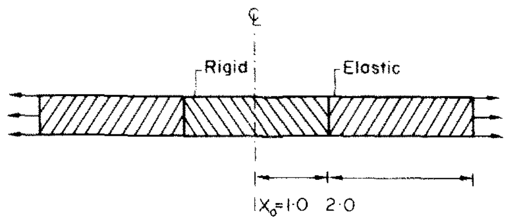

FIG. 2. Section of the bonded circular plate.

plate. The material is assumed to be of Mooney type with $\Gamma = 0{\cdot}1$. The radial displacement u and the radial stress resultant n_x are given in Table 2. It is interesting to note that the radial displacement is approximately constant throughout the plate.

Next, the solution state of an annular circular plate which is bonded to a rigid inclusion at its inner edge and subjected to uniform radial tension at its outer edge is given (Fig. 2). The solution state is assumed to consist of an inner solution at the inner edge, and an outer solution. In Tables 3 and 4 the variation of the radial displacement, the radial and circumferential stress resultants and the extension ratios of the outer problem are given. The variation of the boundary-layer quantities with respect to the stretched coordinate τ is given in Tables 5 and 6. The extension ratio λ which represents the variation of the deformed thickness is plotted in Fig. 3.

CONCLUDING REMARKS

A close look at the second problem reveals that away from the bonded edge the thickness of the deformed plate is again almost constant. Near the bonded edge, where the inner solution is valid, the circumferential stress resultant n_θ shows a sharp rise (Tables 3 and 5) its value at the bonded edge ($\tau = 0$) being almost three times larger than the one predicted

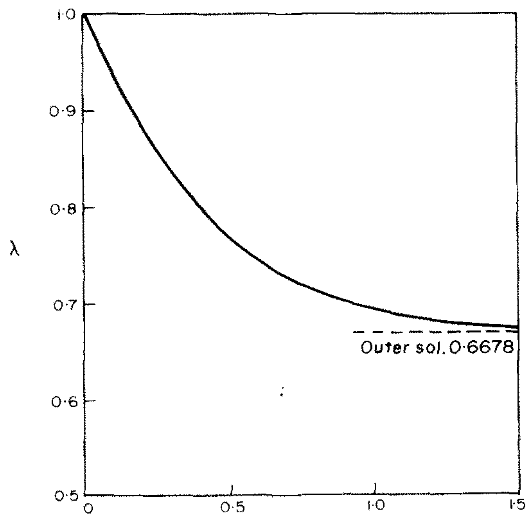

FIG. 3. Variation of the extension ratio λ.

TABLE 5. BONDED PLATE INNER SOLUTION: STRESS RESULTANTS AND COUPLES
($\varepsilon = 0{\cdot}04166$, $\Gamma = 0{\cdot}1$)

τ	x	$-s_x$	n	n_θ
0·00	1·0000	0·7793	1·8682	1·3662
0·25	1·0104	0·4208	0·9134	1·0010
0·50	1·0208	0·2202	0·4489	0·7627
0·75	1·0312	0·1124	0·2211	0·6189
1·00	1·0417	0·0565	0·1090	0·5385
1·25	1·0521	0·0281	0·0538	0·4959
1·50	1·0625	0·0139	0·0265	0·4741
∞	—	0·0000	0·0000	0·4523

by the outer solution. The thickness of the stretched plate, characterized by the extension ratio λ, shows an exponential decay to its asymptotic value found from the outer solution.

It can be concluded that the effect of the symmetric transverse normal strain is localized near the edges of the plate, but can affect significantly the stress concentration around the bonded edges.

TABLE 6. BONDED PLATE INNER SOLUTION: EXTENSION RATIOS

τ	λ_x	λ
0·00	1·0000	1·0000
0·25	1·1666	0·8573
0·50	1·2978	0·7706
0·75	1·3857	0·7217
1·00	1·4376	0·6956
1·25	1·4659	0·6822
1·50	1·4806	0·6754
∞	1·4954	0·6687

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Абстракт—Исследуется задача конечного растяжения кольцевой пластинки, связанной с жестким включением на ее внутреннем краю. Рассматривается материал изотропный и несжимаемый, с конститутивном закономмциа Мунзя. Указано, что включение эффекта поперечной нормальной деформации ведёт к быстрому изменению толщины, ограниченной узкой краеьой зоной. Даются решения, в явном виде для уравнений пограничного слоя, представляющих поведение пластинки вблизи краев.